

# Some intersection numbers of divisors on toroidal compactifications of $\mathcal{A}_g$

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## Abstract

We study the top intersection numbers of the boundary and Hodge class divisors on toroidal compactifications of the moduli space  $\mathcal{A}_g$  of principally polarized abelian varieties and compute those numbers that live away from the stratum which lies over the closure of  $\mathcal{A}_{g-3}$  in the Satake compactification.

## 1 Introduction

The moduli space  $\mathcal{A}_g$  of principally polarized abelian varieties of genus  $g$  has been an object of investigation for many years. It is well known that  $\mathcal{A}_g$  admits various compactifications, notably the Satake, sometimes also called the minimal compactification  $\mathcal{A}_g^{\text{Sat}}$ , as well as various toroidal compactifications. The latter depend on the choice of an admissible fan such as the second Voronoi decomposition, the perfect cone or first Voronoi decomposition, or the central cone decomposition. Due to the recent work of several authors, most notably V. Alexeev and N. Shepherd-Barron, it has now become more clear which role the various compactifications play. Alexeev [Al] has shown that the Voronoi compactification  $\mathcal{A}_g^{\text{Vor}}$ , given by the second Voronoi decomposition, solves a moduli problem. More precisely, he has shown that it is an irreducible component of a natural modular compactification, and the situation has been further clarified by Olsson [Ol]. Shepherd-Barron [S-B] proved that the perfect cone compactification  $\mathcal{A}_g^{\text{Perf}}$

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is a canonical model in the sense of Mori theory, provided  $g \geq 12$ . Finally, the central cone decomposition  $\mathcal{A}_g^{\text{Centr}}$  coincides with Igusa's partial desingularization  $\mathcal{A}_g^{\text{Igu}}$  of the Satake compactification  $\mathcal{A}_g^{\text{Sat}}$ . The various toroidal compactifications coincide for  $g \leq 3$ , but are all different in general.

The perfect cone compactification has a very simple Picard group: it is generated (over  $\mathbb{Q}$ ) by the determinant of the Hodge bundle  $L$  and the (irreducible) boundary divisor  $D$ . In this paper we need to be careful to distinguish the computations on the stack and on the coarse moduli space: a generic ppav has an involution  $-1$ , and thus a generic point of  $\mathcal{A}_g$  has a stabilizer of order two. Thus let us specify explicitly that by  $D$  we mean the substack  $\mathcal{A}_g^{\text{Perf}} \setminus \mathcal{A}_g$  of  $\mathcal{A}_g^{\text{Perf}}$ .

Shepherd-Barron [S-B, p. 41] has posed the question of determining the top intersection numbers of divisors

$$a_N^{(g)} := \langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\text{Perf}}} \quad (*)$$

where  $G := \frac{g(g+1)}{2}$  is the dimension of  $\mathcal{A}_g^{\text{Perf}}$ . It is this question which we want to address in this paper and to which we give a partial answer:

**Theorem 1.1.** *The only three intersection numbers with  $N < 3g - 3$  that are non-zero are those for  $N = 0, g, 2g - 1$  (and thus the power of  $L$  being equal to  $\dim \mathcal{A}_g, \dim \mathcal{A}_{g-1}$ , and  $\dim \mathcal{A}_{g-2}$ , respectively). The numbers are*

$$\langle L^{\frac{g(g+1)}{2}} \rangle_{\mathcal{A}_g^{\text{Perf}}} = (-1)^G 2^{-g} G! \prod_{k=1}^g \frac{\zeta(1-2k)}{(2k-1)!!} \quad (1)$$

$$\langle L^{\frac{(g-1)g}{2}} D^g \rangle_{\mathcal{A}_g^{\text{Perf}}} = \frac{1}{2} (-1)^{G-1} (g-1)! (G-g)! \prod_{k=1}^{g-1} \frac{\zeta(1-2k)}{(2k-1)!!} \quad (2)$$

and

$$\langle L^{\frac{(g-2)(g-1)}{2}} D^{2g-1} \rangle_{\mathcal{A}_g^{\text{Perf}}} = (I) + (II) + (III) \quad (3)$$

where the terms (I), (II) and (III) are given explicitly by Theorems 9.5, 8.3 and 7.5 respectively.

The most striking result of our computations is that

$$\langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\text{Perf}}} = 0 \quad \text{unless} \quad G - N = \dim \mathcal{A}_k \quad \text{for some } k \leq g \quad (4)$$

in the range  $N < 3g - 3$ . This leads one naturally to

**Conjecture 1.2.** *The intersection numbers  $a_N^{(g)}$  for any  $N$  vanish unless  $G - N = k(k + 1)/2$  for some  $k \leq g$ .*

One can also ask this question for other toroidal compactifications of  $\mathcal{A}_g$ , and it is tempting to conjecture that if one interprets  $D$  as the closure of the boundary of the partial compactification, this would also hold.

Of course, one could even hope that such a vanishing result holds for perfect cone compactifications or in fact for all (reasonable) toroidal compactifications of any quotient of a homogeneous domain by an arithmetic group. This is e.g. the case for the moduli space of polarized K3 surfaces. However, in this case the minimal compactification has only two boundary strata, which are of dimension 0 and 1 respectively. Thus the vanishing is essentially automatic from our discussion in Section 2.

Of the three non-zero numbers above, the top self-intersection number  $a_0^{(g)} = L^G$  of the Hodge line bundle  $L$  can be computed using Hirzebruch-Mumford proportionality, see e.g. [vdG1, Theorem 3.2]. The second non-zero intersection number can be easily obtained following the methods of van der Geer from [vdG2], in working on the partial compactification — we give the details in the next section. Thus in the theorem above, apart from the vanishing results, the significant new non-zero number we compute is  $a_{2g-1}^{(g)} = \langle L^{\frac{(g-2)(g-1)}{2}} D^{2g-1} \rangle_{\mathcal{A}_g^{\text{Perf}}}$ .

The intersection theory on  $\mathcal{A}_g^{\text{Perf}}$  for  $g \leq 3$  is described completely (including the intersections of higher-dimensional classes) by van der Geer [vdG2], while the intersection theory of divisors for the perfect cone compactification as well as for the more complicated second Voronoi compactification for  $g = 4$  was computed in our previous work [EGH]. We have greatly benefited from van der Geer's methods in [vdG1], [vdG2].

Throughout the paper we work with  $\mathcal{A}_g^{\text{Perf}}$ , but our computations also provide information for other toroidal compactifications (see Section 11).

Throughout this paper we work over the field of complex numbers  $\mathbb{C}$ .

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## 2 Intersections on the partial compactification

We shall now compute the intersection numbers  $a_N^{(g)}$  for  $N \leq 2g - 2$ . The considerations of this section are true for all toroidal compactifications  $\overline{\mathcal{A}}_g$ .

We have already remarked that the number  $L^G = a_0^{(g)}$  can be computed via Hirzebruch-Mumford proportionality. All other intersection numbers involve the boundary and hence it is essential to obtain a good understanding of its geometric structure. Every toroidal compactification  $\overline{\mathcal{A}}_g$  admits a projection

$$\pi : \overline{\mathcal{A}}_g \rightarrow \mathcal{A}_g^{\text{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \dots \sqcup \mathcal{A}_0 \quad (5)$$

to the Satake compactification.

Following [vdG2] we denote  $\beta_k := \pi^{-1}(\mathcal{A}_{g-k}^{\text{Sat}})$ . We also denote  $\mathcal{A}_g^{\text{part}} := \overline{\mathcal{A}}_g \setminus \beta_2$ . This is the partial compactification considered by Mumford [Mu1], and it is independent of the choice of the toroidal compactification. By abuse of notation we shall also use the symbol  $D$  instead of  $\beta_1 \setminus \beta_2$  if it is clear that we are working away from  $\beta_2$ . The boundary of the partial compactification is the universal Kummer family over  $\mathcal{A}_{g-1}$ . More precisely, let  $\mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$  be the universal abelian variety of dimension  $g-1$  (which exists as a stack), then there exists a degree 2 map  $j : \mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$  and thus (cf. [vdG1, p. 5])

$$j_*([\mathcal{X}_{g-1}]) = 2D. \quad (6)$$

Note that  $D$  is denoted by  $\sigma_1$  in [vdG1]. In computing the intersection numbers we have to be careful to distinguish between stacks and varieties. We will, however, not need the full force of the stacky machinery as all our calculations can be done by going to suitable level covers.

In order to compute intersection numbers involving the boundary, first note that  $D \in \text{CH}^*(\beta_1)$ . Here  $\text{CH}^*$  denotes the Chow ring (tensored with  $\mathbb{Q}$ , since torsion is irrelevant for the intersection numbers). Hence any intersection numbers involving  $D$  can be computed on  $\beta_1$ .

The following lemma is well-known, but we include the proof for easy reference.

**Lemma 2.1.** *For every toroidal compactification  $\overline{\mathcal{A}}_g$  we have*

$$L^M|_{\beta_k} = 0 \in \text{CH}^*(\beta_k) \quad \text{for} \quad M > \frac{(g-k)(g-k+1)}{2}. \quad (7)$$

*In particular*

$$a_N^{(g)} = 0 \quad \text{for } 1 \leq N \leq g-1.$$

*Proof.* Recall that suitable powers of  $L$  are globally generated and can be used to define the Satake compactification  $\mathcal{A}_g^{\text{Sat}}$  as the closure of the image of  $\mathcal{A}_g$  under the morphism defined by  $L^{\otimes k}$ . Taking general hyperplane sections we can thus represent (a multiple of)  $L^N$  in  $\mathcal{A}_g^{\text{Sat}}$  by a cycle which has empty

intersection with  $\mathcal{A}_{g-k}^{\text{Sat}}$ , as this space has dimension  $(g-k)(g-k+1)/2$ . This gives the first claim (cf. also the proof of [vdG1, Lemma 1.2]).

The second claim follows since  $D \in \text{CH}^*(\beta_1)$  and  $L^{G-N}|_{\beta_1} = 0$  for  $N > (g-1)g/2$ .  $\square$

In the range  $N \geq 2g-2$  we know that  $L^{G-N}|_{\beta_2} = 0$ . By the inclusion Chow exact sequence

$$\text{CH}^*(\beta_2) \rightarrow \text{CH}^*(\beta_1) \rightarrow \text{CH}^*(\beta_1 \setminus \beta_2) \rightarrow 0$$

the intersection numbers  $a_N^{(g)}$  for  $g \leq N \leq 2g-2$  can be computed on the stratum  $\beta_1 \setminus \beta_2$ , i.e. on the universal Kummer family.

It is well known (cf. [Mu1, Proposition 1.8] and [vdG1, Lemma 1.1]) that the pullback of the boundary divisor  $D$  under the map  $j : \mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$  is given by

$$j^*D = -2\Theta \tag{8}$$

where  $\Theta$  denotes the symmetric theta divisor on  $\mathcal{X}_{g-1}$  trivialized along the 0-section. Taking the symmetric theta divisor means taking the divisor of the universal theta function and dividing it by the theta constant. Since the theta constant is a modular form of weight  $1/2$ , i.e. a section of  $L/2$ , this means that  $\Theta = \Theta' - L/2$  where  $\Theta'$  is the divisor of the universal theta function used by Mumford in [Mu1].

Following Mumford's ideas from [Mu1], it was shown by van der Geer [vdG2, Formula (3)] that for the universal family  $\pi : \mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$  the pushforwards of the powers of  $\Theta$  are

$$\pi_*(\Theta^{g-1}) = (g-1)![\mathcal{A}_{g-1}] \quad \text{and} \quad \pi_*(\Theta^{g+\varepsilon}) = 0 \quad \forall \varepsilon \geq 0. \tag{9}$$

From this one easily obtains the following intersection numbers:

**Proposition 2.2.** *We have*

- (i)  $a_g^{(g)} = \frac{1}{2}(-2)^{g-1}(g-1)!a_0^{(g-1)} = \frac{1}{2}(-1)^{G-1}(g-1)!(G-g)! \prod_{k=1}^{g-1} \frac{\zeta(1-2k)}{(2k-1)!}$
- (ii)  $a_{g+1}^{(g)} = \dots = a_{2g-2}^{(g)} = 0$ .

*Proof.* We have already observed that  $L^{G-N}|_{\beta_2} = 0 \in \text{CH}^*(\beta_2)$  in the range  $g \leq N < 2g-1$ . Using the Chow inclusion exact sequence we obtain

$$a_N^{(g)} = \langle L^{G-N} D^N \rangle_{\overline{\mathcal{A}}_g} = \frac{1}{2} \langle L^{G-N} j^*(D^{N-1}) \rangle_{\mathcal{X}_{g-1}}$$

$$= \frac{1}{2} \langle L^{G-N} \pi_*((-2\Theta)^{N-1}) \rangle_{\mathcal{A}_{g-1}}$$

where the factor  $1/2$  in the first equality comes from equation (6) and where we have used equation (8) and the projection formula in the last equality. However, we know from (9) that  $\pi_*(\Theta^{g+\varepsilon}) = 0 \in \mathrm{CH}^*(\mathcal{A}_{g-1})$  for all  $\varepsilon \geq 0$ . Since the intersection numbers in question can be computed on  $\beta_2 \setminus \beta_1$ , this gives claim (ii).

To obtain (i) we compute

$$\begin{aligned} a_g^{(g)} &= \frac{1}{2} (-2)^{g-1} \langle L^{\frac{(g-1)g}{2}} \pi_*(\Theta^{g-1}) \rangle_{\mathcal{A}_{g-1}} \\ &= \frac{1}{2} (-2)^{g-1} (g-1)! \langle L^{\frac{(g-1)g}{2}} \rangle_{\mathcal{A}_{g-1}} = \frac{1}{2} (-2)^{g-1} (g-1)! a_0^{(g-1)} \end{aligned}$$

and thus claim (i) follows from Hirzebruch-Mumford proportionality for genus  $g-1$ .  $\square$

**Remark 2.3.** These intersection numbers have been computed in a different way for  $g \leq 4$  in [vdG2] and [EGH], and our results agree with those — the extra factors of 2 in comparison with [EGH] are due to the fact that we work with the stack  $\mathcal{A}_g$  as compared to the variety as we did in [EGH].

For example, for  $g = 4$  we get

$$-\frac{1}{7560} = \langle L^6 D^4 \rangle_{\overline{\mathcal{A}}_4} = \frac{1}{2} (-2)^3 3! \langle L^6 \rangle_{\mathcal{A}_3} = -24 \cdot \frac{1}{181440}.$$

### 3 Geometry of the locus of corank $\leq 2$ degenerations

From now on we shall work with the perfect cone compactification  $\mathcal{A}_g^{\mathrm{Perf}}$ . We have already remarked that there is a map  $j : \mathcal{X}_{g-1} \rightarrow \mathcal{A}_g^{\mathrm{part}}$  which maps the universal family  $2 : 1$  onto the boundary. It is crucial for us that this picture of the boundary can be extended one step further. More precisely, the universal family  $j : \mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$  has a partial compactification  $\mathcal{X}_{g-1}^{\mathrm{part}} \rightarrow \mathcal{A}_{g-1}^{\mathrm{part}}$  such that the map  $j : \mathcal{X}_{g-1} \rightarrow \mathcal{A}_g^{\mathrm{part}}$  extends to a map  $j : \mathcal{X}_{g-1}^{\mathrm{part}} \rightarrow \mathcal{A}_g^{\mathrm{Perf}}$  with image  $j_*[\mathcal{X}_{g-1}^{\mathrm{part}}] = 2[\beta_1 \setminus \beta_3]$ .

For details of the geometry (of the underlying variety) of the stack  $\mathcal{X}_{g-1}^{\mathrm{part}}$  see [Ts], [Hu]. Here we recall the essential properties of this stack. The fibration  $\pi : \mathcal{X}_{g-1}^{\mathrm{part}} \rightarrow \mathcal{A}_{g-1}^{\mathrm{part}}$  extends the universal family of genus  $g-1$ . The fiber over a point in  $\mathcal{X}_{g-2} = \mathcal{A}_{g-1}^{\mathrm{part}} \setminus \mathcal{A}_{g-1}$  is a corank 1 degeneration

of a  $(g - 1)$ -dimensional abelian variety, i.e. a  $\mathbb{P}^1$ -bundle over an abelian variety of genus  $g - 2$  where the 0-section and the  $\infty$ -section are glued with a shift. We denote the composition of the projection  $\mathcal{X}_{g-1}^{\text{part}} \rightarrow \mathcal{X}_{g-2}$  with the projection  $\mathcal{X}_{g-2} \rightarrow \mathcal{A}_{g-2}$  by  $\pi_{\text{Sat}}$ . Then we have the following picture: let  $B \in \mathcal{A}_{g-2}$  and let  $\mathcal{P}_B$  denote the Poincaré line bundle on  $B \times B$ . We denote the coordinates on  $B \times B$  by  $(z, b)$ . Note that the coordinates are *not* interpreted symmetrically — the first one defines an actual point on  $B$ , while the second is thought of as parameterizing the moduli of semiabelian varieties with abelian part  $B$ . Then

$$\pi_{\text{Sat}}^{-1}(B) = \mathbb{P}^1(\mathcal{P}_B \oplus \mathcal{O}) / (z, b, 0) \sim (z + b, b, \infty). \quad (10)$$

For a discussion of this see also [Mu1, p. 356]. We would like to point out, however, that in our convention the role of the two factors is interchanged compared to [Mu1].

The map  $\pi : \mathcal{X}_{g-1}^{\text{part}} \rightarrow \mathcal{A}_{g-1}^{\text{part}}$  is no longer smooth, but drops rank at the singular locus of the degenerate abelian varieties. We denote this locus, which will be crucial to our considerations, by  $\Delta$ . Note that  $\Delta = \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \hat{\mathcal{X}}_{g-2}$  (see e.g. [vdG2, p. 11]). Using the principal polarization we can identify this with  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$ , but we will be careful to usually not perform this identification, as the two factors in  $\Delta$  geometrically play a very different role. Clearly  $\Delta$  is of codimension two in  $\mathcal{A}_g^{\text{Perf}}$ . As a stack it is smooth.

We are also interested in the restriction of the above family to the boundary, i.e. in the family  $\pi : \mathcal{X}_{g-1}^{\text{part}} \setminus \mathcal{X}_{g-1} \rightarrow \mathcal{X}_{g-2}$ . The total space is now no longer smooth, in fact it is a non-normal with singular locus  $\Delta$ . Locally along  $\Delta$  we have two smooth divisors intersecting transversally. If we remove  $\Delta$  from  $\mathcal{X}_{g-1}^{\text{part}}$  we obtain a  $\mathbb{C}^*$ -fibration over  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \hat{\mathcal{X}}_{g-2}$  whose total space can be identified with the Poincaré bundle  $\mathcal{P}$  with the 0-section removed. If  $Y$  is the normalization of  $\mathcal{X}_{g-1}^{\text{part}}$ , then this implies that  $Y = \mathbb{P}^1(\mathcal{P} \oplus \mathcal{O})$ . We must keep track of the various projection maps involved and therefore want to give them names. The geometry is summarized by the following

commutative diagram of projection maps:

$$\begin{array}{ccccc}
 Y = & & \mathbb{P}^1(\mathcal{P} \oplus \mathcal{O}) & & \\
 & \searrow & \downarrow f & & \\
 \Delta = & \tilde{\pi} & \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \hat{\mathcal{X}}_{g-2} & & \\
 & \swarrow pr_1 & \downarrow h & \searrow pr_2 & \\
 \mathcal{X}_{g-2} & & \mathcal{A}_{g-2} & & \hat{\mathcal{X}}_{g-2} \\
 & \searrow p_2 & & \swarrow p_1 & \\
 & & \mathcal{A}_{g-2} & & 
 \end{array} \tag{11}$$

Here the notation for the map  $\tilde{\pi}$  is due to the fact that it is the composition of the normalization map  $Y \rightarrow \mathcal{X}_{g-1}^{\text{part}}$  with the restriction of the universal family  $\pi : \mathcal{X}_{g-1}^{\text{part}} \rightarrow \mathcal{A}_{g-1}^{\text{part}}$  to the boundary  $\mathcal{X}_{g-2}$  of  $\mathcal{A}_{g-1}^{\text{part}}$ .

## 4 Intersection theory on $\Delta$ and $Y$

In the beginning of this section the geometry of  $\Delta$  sitting inside  $\mathcal{A}_g^{\text{Perf}}$  will not be important for us, and thus we will for now use the principal polarization to identify  $\Delta = \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$ . We first want to understand the Néron-Severi group of  $\mathbb{Q}$ -divisors on  $\Delta$  modulo numerical equivalence. Let  $T_i = pr_1^*(\Theta)$  for  $i = 1, 2$  where  $pr_i$  is the  $i$ -th projection from  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$  to  $\mathcal{X}_{g-2}$  (as in the diagram above), and let  $\mathcal{P}$  be the Poincaré bundle on  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$ .

**Lemma 4.1.** *For  $g \geq 4$  the Néron-Severi group of  $\mathbb{Q}$ -divisors on  $\Delta$  modulo numerical equivalence has rank 4; more precisely*

$$\text{NS}_{\mathbb{Q}}(\Delta) = \text{NS}_{\mathbb{Q}}(\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}) = \mathbb{Q}h^*L \oplus \mathbb{Q}T_1 \oplus \mathbb{Q}T_2 \oplus \mathbb{Q}\mathcal{P}.$$

*If  $g = 3$  then  $\text{NS}_{\mathbb{Q}}(\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2})$  has rank 3 and is generated by  $T_1, T_2$  and  $\mathcal{P}$ .*

*Proof.* We first remark that for the basis of the fibration  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2} \rightarrow \mathcal{A}_{g-2}$  we have  $\text{NS}_{\mathbb{Q}}(\mathcal{A}_{g-2}) = \mathbb{Q}L$ , provided  $g \geq 4$ , otherwise it is trivial. We also observe that for a very general fiber  $A \times A$  (i.e. outside a countable union of proper subvarieties) the rank of  $\text{NS}_{\mathbb{Q}}(A \times A)$  equals 3 [BL, Chapter 5.2]. The fibration  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2} \rightarrow \mathcal{A}_{g-2}$  is topologically locally trivial.



If  $M$  is a line bundle on  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$  then we can write

$$c_1(M) = c_0 h^* L + c_1 [T_1] + c_2 [T_2] + c_3 [\mathcal{P}] + \sum_{i=4}^{2\binom{2g-4}{2}} c_i [\lambda_i]$$

where the  $c_i$  are rational numbers and the  $\lambda_i \in H^2(A \times A, \mathbb{Q})$  are chosen such that they form a basis of  $H^2(A \times A, \mathbb{Q})$  together with  $[T_1]$ ,  $[T_2]$  and  $[\mathcal{P}]$ .

The coefficients of this sum are locally constant. Restricting to a very general fiber, we see that  $c_i = 0$  for  $i \geq 4$ .

For an alternative proof see [S-B, Lemma 2.10].  $\square$

We will now want to understand the intersection theory of divisors on  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$ . Much of what we need was done in [GL], where one fiber of this universal family was studied. Since we need some more information and our notations are different, we review and extend the setup.

We have a shift operator

$$s : \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2} \rightarrow \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$$

$$s(\tau, z, b) := (\tau, z + b, b)$$

where we will use notations  $z$  and  $b$  for the variables on the first and second factor of  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$ , respectively. It is useful to keep in mind that geometrically  $b$  lies on the dual abelian variety.

**Lemma 4.2** ([GL]). *The shift operator acts on  $\mathrm{NS}_{\mathbb{Q}}(\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2})$  by*

$$(i) \quad s^* T_1 = T_1 + T_2 + \mathcal{P}$$

$$(ii) \quad s^* T_2 = T_2$$

$$(iii) \quad s^* \mathcal{P} = 2T_2 + \mathcal{P}$$

$$(iv) \quad s^*(h^* L) = h^* L.$$

*Proof.* Equality (iv) is obvious, since  $h \circ s = h$ . We shall prove this lemma by intersecting with suitable test curves. For this let  $A \in \mathcal{A}_{g-2}$  be a general ppav, and let  $C \subset A$  be a general curve of degree  $n := \langle C \Theta_A \rangle_A$  (where  $\Theta_A$  denotes the theta divisor on  $A$ ); denote by  $i_1, i_2$ , and  $i_d$  the three inclusions  $A \hookrightarrow A \times A$  given by mapping to the first factor, to the second factor, and to the diagonal respectively. Moreover let  $C_L \subset \mathcal{A}_{g-2}$  be a generic curve, which we think of as sitting in the 0-section of  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2} \rightarrow \mathcal{A}_{g-2}$ . The

intersection matrix of these curves with the generators of  $\text{NS}_{\mathbb{Q}}(\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2})$  is given by

divisor \ curve	$C_L$	$C_1$	$C_2$	$C_d$
$h^*L$	$\langle C_L L \rangle_{\mathcal{A}_{g-2}}$	0	0	0
$T_1$	0	$n$	0	$n$
$T_2$	0	0	$n$	$n$
$\mathcal{P}$	0	0	0	$2n$

(12)

The intersection of  $h^*L$  with the curves  $C_1$ ,  $C_2$  and  $C_d$  is clearly 0 since these curves lie in a fiber of the map  $h : \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2} \rightarrow \mathcal{A}_{g-2}$ . The intersections of  $T_i$  with  $C_1, C_2$ , and  $C_d$  are obvious. The intersection of  $T_i$  and  $\mathcal{P}$  with  $C_L$  is 0 since these bundles are, by definition, trivial along the 0-section which contains  $C_L$ . Lastly, we note that the Poincaré bundle is trivial on  $A \times \{0\}$  and  $\{0\} \times A$ , and thus trivial on  $C_1$  and  $C_2$ ; however, on the diagonal the Poincaré bundle restricts to  $\mathcal{O}_A(2\Theta_A)$  (see [Mu1], p. 357). The intersection matrix above is non-singular; hence the curves  $C_1, C_2, C_d$  and  $C_L$  can serve as test curves for the intersection theory on  $\mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \mathcal{X}_{g-2}$ .

We shall now calculate the intersection of  $(s^N)^*(T_1)$  with the test curves where  $N \geq 1$  is an integer. Recall that  $T_1$  is given by the pullback of the divisor  $\{\Theta(\tau, z)/\Theta(\tau, 0) = 0\}$  on  $\mathcal{X}_{g-2}$  via  $pr_1$ . Here  $\Theta(\tau, z) = \Theta_{00}(\tau, z)$  is the standard theta function and the denominator  $\Theta(\tau, 0)$  ensures that  $T_1$  is trivial along the 0-section. Applying the shift operator  $N$  times means that we have to consider the zeroes of the function  $\Theta(\tau, z + Nb)/\Theta(\tau, Nb) = 0$ . Note that this is still trivial along the 0-section, hence the intersection with  $C_L$  remains trivial. To compute the intersection with the curves  $C_i$  for  $i = 1, 2, d$  the denominator is irrelevant as these curves are contained in a fixed fiber. We claim that the number of zeroes of  $\Theta(\tau, z + Nb)$  on  $C_1, C_2$  and  $C_d$  is  $n, N^2n$  and  $(N+1)^2n$  respectively. The first number follows since  $C_1$  is contained in  $b = 0$  and hence the number of zeroes does not change. For  $C_2$  we have to compute the number of zeroes of  $\Theta(\tau, Nb)$  where the variable is now  $b$ . The claim follows since multiplication by  $N$  induces multiplication by  $N^2$  on the second cohomology of an abelian variety, and hence the intersection number is multiplied by  $N^2$ . For  $C_d$  we notice that we have to put  $z = b$  and the claim for this curve now follows in the same way. From these intersection numbers we deduce that  $(s^N)^*(T_1) = T_1 + N^2T_2 + N\mathcal{P}$ . Putting  $N = 1$  gives claim (i). Claim (ii) follows since the shift operator  $s$  leaves the second variable unchanged.

To prove (iii) we can use the formulae  $(s^2)^*(T_1) = T_1 + 4T_2 + 2\mathcal{P}$  (put  $N = 2$ ) and  $(s^2)^*(T_1) = s^*(T_1 + T_2 + \mathcal{P}) = (T_1 + T_2 + \mathcal{P}) + T_2 + s^*\mathcal{P}$ . Comparing these two formulae gives (iii).  $\square$

We now return to the discussion of the geometry of  $\mathcal{X}_{g-1}^{\text{part}}$  and its normalization  $Y = \mathbb{P}(\mathcal{P} \oplus \mathcal{O})$ . We shall need to know the normal bundle of  $\Delta$  in  $\mathcal{X}_{g-1}^{\text{part}}$ .

**Proposition 4.3.** *The normal bundle*

$$N_{\Delta/\mathcal{X}_{g-1}^{\text{part}}} = \mathcal{P} \oplus (\mathcal{P}^{-1} \otimes T_2^{-2})$$

i.e.  $\Delta^2 = \Delta|_{\Delta} = -\mathcal{P} \cdot (\mathcal{P} + 2T_2)|_{\Delta}$  in  $\text{CH}^*(\mathcal{X}_{g-1}^{\text{part}})$ .

*Proof.* For smooth embeddings  $X \subset Y \subset Z$  we have

$$0 \rightarrow N_{X/Y} \rightarrow N_{X/Z} \rightarrow N_{Y/Z}|_X \rightarrow 0$$

and thus the normal bundle of an intersection of two varieties is the sum of their normal bundles. In our situation the total space  $\mathcal{X}_{g-1}^{\text{part}}$  is smooth, containing  $\Delta$ , which is also smooth (all of this in a stack sense). Locally,  $\Delta$  is a local complete intersection of the two branches of  $Y$  glued along the 0-section and the  $\infty$ -section. Thus the normal bundle of  $\Delta$  is the sum of  $N_{0\text{-section}/Y}$  and of the pullback of  $N_{\infty\text{-section}/Y}$  under the shift which controls the gluing of the 0-section and the  $\infty$ -section. The normal bundle to the 0-section is just the bundle we have over it, i.e. the Poincaré bundle. This gives the first summand. We can reverse the role of the 0-section and the  $\infty$ -section by considering  $\mathbb{P}(\mathcal{O} \oplus \mathcal{P}^{-1})$  instead of  $\mathbb{P}(\mathcal{P} \oplus \mathcal{O})$  which gives the second summand. The normal bundle is thus a direct sum of  $\mathcal{P}$  and  $s^*(\mathcal{P}^{-1}) = \mathcal{P}^{-1} \otimes T_2^{-2}$  where the last equality now follows from Lemma 4.2.

The second claim then follows immediately from the double point formula.  $\square$

For what follows we need to understand the intersection theory of the  $\mathbb{P}^1$ -bundle  $Y$ . We will denote the class of the 0-section by  $\xi \in \text{NS}_{\mathbb{Q}}(Y)$ . Then the class of the  $\infty$ -section is  $\xi - f^*\mathcal{P} \in \text{NS}_{\mathbb{Q}}(Y)$ .

**Lemma 4.4.** *The group of divisors on  $Y$  up to numerical equivalence is  $\text{NS}_{\mathbb{Q}}(Y) = f^*\text{NS}_{\mathbb{Q}}(\Delta) \oplus \mathbb{Q}\xi$ . Moreover*

$$\xi(\xi - f^*\mathcal{P}) = 0 \tag{13}$$

in  $\text{CH}^*(Y)$ .

*Proof.* Clearly  $\text{NS}_{\mathbb{Q}}(Y)$  is generated by  $f^*\text{NS}_{\mathbb{Q}}(\Delta)$  together with the class  $\xi$ . The relation  $\xi(\xi - f^*\mathcal{P}) = 0$  is then simply the fact that the 0-section and the  $\infty$ -section do not intersect. See also [Fu, Remark 3.24].  $\square$

**Proposition 4.5** (see [GL]). *The pullback of the symmetric theta divisor  $\Theta$  to  $Y$  is numerically*

$$\Theta|_Y = \xi + f^*T_1 - \frac{1}{2}f^*\mathcal{P} \in \text{NS}_{\mathbb{Q}}(Y). \quad (14)$$

*Proof.* We refer to [GL] for a more general discussion of divisor classes on the universal semiabelian family, possibly with a level structure. Since we are interested in the universal symmetric theta divisor (as opposed to the universal theta function), for completeness we give a full proof below. We prove this result again by computing the intersections of the divisor classes with suitable test curves.

We use the curves  $C_1$ ,  $C_2$ ,  $C_d$  and  $C_L$  as in the proof of Lemma 4.2, but this time we embed them into the  $\infty$ -section of  $Y$ . As a fifth test curve we consider a general fiber  $F$  of the  $\mathbb{P}^1$ -bundle  $Y$ . We claim that the 5 curves described are a suitable set of test curves. Indeed, this follows from the following intersection matrix

divisor \ curve	$C_L$	$C_1$	$C_2$	$C_d$	$F$
$f^*h^*L$	$\langle C_L L \rangle_{\mathcal{A}_{g-2}}$	0	0	0	0
$f^*T_1$	0	$n$	0	$n$	0
$f^*T_2$	0	0	$n$	$n$	0
$f^*\mathcal{P}$	0	0	0	$2n$	0
$\xi$	0	0	0	0	1

(15)

The top left hand  $4 \times 4$  matrix follows immediately from Lemma 4.2. The intersection of the divisors  $f^*h^*L$ ,  $f^*T_1$ ,  $f^*T_2$  and  $f^*\mathcal{P}$  with  $F$  is clearly 0 since these curves are contracted by  $f$ . Similar the intersection of  $\xi$  with these curves is 0 since  $\xi$  is the 0-section and these curves lie in the  $\infty$ -section. The remaining number  $\xi.F = 1$  is also obvious.

We now want to compute the intersection numbers of the symmetric theta divisor with these test curves. The symmetric theta divisor on the universal semiabelian variety is given by the function

$$\Theta_{\text{symm}} = \frac{\theta(\tau, z + b/2) + x\theta(\tau, z - b/2)}{2\theta(\tau, b/2)}$$

where  $\theta(\tau, z) = \Theta_{00}(\tau, z)$  is the standard theta function. Here  $x$  is the variable on the  $\mathbb{P}^1$  fiber of  $Y$  over  $\Delta$ . Note that the denominator ensures that the corresponding divisor is trivial on the section given by the neutral element of the universal semiabelian variety, since setting  $z = 0$  and  $x = 1$  gives the constant 1 (see also [HuWe]). Clearly the degree of  $\Theta_{\text{symm}}$  on  $F$

is 1. The curves  $C_1$ ,  $C_2$ ,  $C_d$  and  $C_L$  all lie in the  $\infty$ -section of  $Y$ . To evaluate  $\Theta_{\text{symm}}$  on these curves therefore means that we have to consider only the second summand  $\Theta(\tau, z - b/2)/2\Theta(\tau, b/2)$ . To compute the intersection with  $C_L$  we have to set  $z = 0$  and we immediately find that this degree is 0. The same argument applies to  $C_2$ . For  $C_1$  we must put  $b = 0$  and we see that the degree equals  $n$  (recall that our computations happen on a general abelian variety, and in particular that we can assume  $\Theta(\tau, 0) \neq 0$ , so that the denominator is non-zero). Finally, in order to compute the intersection number with  $C_d$  we put  $z = b$  and we obtain again 0. The discussion above is summarized by the following intersection numbers:

divisor \ curve	$C_L$	$C_1$	$C_2$	$C_d$	$F$
$\Theta _Y$	0	$n$	0	0	1

(16)

It now follows immediately from these intersection numbers that

$$\Theta|_Y = \xi + f^*T_1 - \frac{1}{2}f^*\mathcal{P}$$

as claimed. □

**Corollary 4.6.** *We have*

$$\Theta|_\Delta = T_1 + \frac{1}{2}\mathcal{P} \in \text{NS}_{\mathbb{Q}}(\Delta)$$

*Proof.* This follows immediately from Proposition 4.5 by restricting to the 0-section since  $\xi^2 = \xi f^*\mathcal{P}$  (Note that we obtain the same result if we first restrict to the  $\infty$ -section and then pull back via  $s$ .) □

## 5 Level cover of the moduli space

In order to do our intersection calculation we want to pass to a suitable level cover of  $\mathcal{A}_g$ . We shall only consider level covers of full level  $\ell$  and we assume  $\ell \geq 3$  to be prime. This simplifies many of the formulae and is clearly sufficient. We denote by  $\mathcal{A}_g^{\text{Sat}}(\ell)$  and  $\mathcal{A}_g^{\text{Perf}}(\ell)$  the Satake and the perfect cone compactification of  $\mathcal{A}_g(\ell)$  respectively. They fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_g^{\text{Perf}}(\ell) & \xrightarrow{\sigma} & \mathcal{A}_g^{\text{Perf}} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{A}_g^{\text{Sat}}(\ell) & \xrightarrow{\sigma} & \mathcal{A}_g^{\text{Sat}} \end{array}$$

where  $\sigma$  is a Galois cover with Galois group  $\mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})$ . The above diagram is interpreted as a diagram of stacks. Note that for level  $\ell \geq 3$  the element  $-1$  is no longer contained in the level- $\ell$  subgroup of  $\mathrm{Sp}(2g, \mathbb{Z})$  and we can, therefore, compute intersection numbers on the level covers directly on the underlying varieties.

We will denote by  $\beta_i(\ell)$  the preimage in  $\mathcal{A}_g^{\mathrm{Perf}}(\ell)$  of  $\mathcal{A}_{g-i}^{\mathrm{Sat}}(\ell)$ . Unlike the no level case, the boundary of  $\mathcal{A}_g^{\mathrm{Perf}}(\ell)$  has many irreducible components, which we denote  $D_i(\ell) = D_i$ . We also recall (cf. [vdG1, p. 4]) that

$$\sigma^*(D) = \sum_i \ell D_i. \quad (17)$$

The geometry of toroidal compactifications is controlled by the admissible fan which is chosen. In the case of  $\mathcal{A}_g$  this is a fan in the space  $\mathrm{Sym}^{\geq 0}(g, \mathbb{R})$  of real semi-positive definite symmetric  $g \times g$  matrices. Here we work with the perfect cone or first Voronoi decomposition, which is defined by taking the convex hull of all primitive rank one forms in  $\mathrm{Sym}^{\geq 0}(g, \mathbb{Z})$ . We want to understand the geometry of  $\mathcal{A}_g^{\mathrm{Perf}}(\ell) \setminus \beta_3(\ell)$ . For this we only need the intersection of the perfect cone decomposition with the subspace  $\mathrm{Sym}^{\geq 0}(2, \mathbb{R}) \subset \mathrm{Sym}^{\geq 0}(g, \mathbb{R})$  where the inclusion is given by taking the first two variables. For  $g \leq 3$  the first and the second Voronoi decomposition coincide, and both decompositions are well understood for  $g \leq 4$ , see [V1], [V2a] and [V2b] as well as [HS] for a more recent exposition. Note that as in the case without level structure

$$D_i \cong \mathcal{X}_{g-1}^{\mathrm{part}}(\ell).$$

We first recall

**Proposition 5.1.** *The following identity holds in  $\mathrm{CH}^*(\mathcal{A}_g^{\mathrm{Perf}}(\ell) \setminus \beta_3(\ell))$ :*

$$D_i|_{D_i} = -2 \frac{\Theta}{\ell} |_{D_i}.$$

Here  $\Theta$  is the pullback of the theta divisor trivialized along the 0-section to the universal family  $\mathcal{X}_{g-1}^{\mathrm{part}}(\ell)$  with level- $\ell$  structure.

*Proof.* This follows from [Hu, Proposition 3.2].  $\square$

We will now discuss the intersection of the boundary components. Recall that the boundary components  $D_i$  correspond to primitive integer quadratic forms of rank 1, i.e. to squares  $q_i(x) = l_i^2(x)$  where the  $l_i$  are primitive integral linear forms. The intersection of two different components  $D_i$  and

$D_j$  is always contained in  $\beta_2(\ell)$ , and two components  $D_i$  and  $D_j$  have a non-empty intersection if and only if the corresponding quadratic forms  $l_i^2$  and  $l_j^2$  are part of a basis of  $\text{Sym}(g, \mathbb{Z})$ . The situation is more complicated for the intersection of three different boundary components  $D_i$ ,  $D_j$  and  $D_k$ . In this case we have two possibilities: the linear forms  $l_i$ ,  $l_j$  and  $l_k$  can either be linearly independent or not. In the first case the intersection of the three boundary components is contained in  $\beta_3(\ell)$ . We will refer to this as the *global* case. We will not be concerned with this as this will not contribute to the intersection numbers. The other case is called the *local* case and we will have to consider these intersections.

The intersection of four boundary divisors will never contribute to the intersection numbers that we will compute. This follows from

**Lemma 5.2.** *For any four distinct indices  $i, j, k, l$  we have*

$$D_i \cap D_j \cap D_k \cap D_l \subset \beta_3(\ell)$$

and thus  $D_i D_j D_k D_l = 0 \in \text{CH}^*(\mathcal{A}_g^{\text{Perf}}(\ell) \setminus \beta_3(\ell))$ .

*Proof.* This follows from the fact that there are no cones in the perfect cone decomposition of  $\text{Sym}^{\geq 0}(2, \mathbb{R})$  which are spanned by 4 vectors.  $\square$

We now want to understand the intersection of divisors more systematically. We will have to compute intersections of the form  $D_i^a D_j^b D_k^c L^d$  such that  $a+b+c+d = G$ . For symmetry reasons these numbers will only depend on the integers  $a$ ,  $b$  and  $c$ . To compute these intersections we will, however, work on a specific boundary component  $D_i$  and thus we must understand how  $D_i|_{D_i \cap D_j}$  and  $D_j|_{D_i \cap D_j}$  compare. Boundary components correspond to rank 1 symmetric forms. As we are only interested in the case where the intersection is of local type, these three forms are linearly dependent. Modulo the action of the group  $\text{GL}(g, \mathbb{Z})$  we can assume that the boundary components  $D_i$ ,  $D_j$  and  $D_k$  correspond to the matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$  respectively. This is shorthand for  $g \times g$ -matrices where these matrices sit in the upper left corner and the rest of the entries are zero. Correspondingly we consider  $\text{GL}(2, \mathbb{Z})$  as a subgroup  $\text{GL}(g, \mathbb{Z})$ . An element  $\gamma \in \text{GL}(g, \mathbb{Z})$  acts on a matrix  $M$  by  $M \mapsto {}^t \gamma^{-1} M \gamma^{-1}$ .

Now consider the element  $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  interpreted as an element in  $\text{GL}(g, \mathbb{Z})$ . This interchanges  $D_i$  and  $D_j$  while it fixes  $D_k$ . In terms of the

symplectic group we are considering the matrix

$$\gamma = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \mathbf{1}_{g-2} & & \\ & & & 0 & 1 \\ & & & 1 & 0 \\ & & & & & \mathbf{1}_{g-2} \end{pmatrix}$$

This matrix acts on Siegel space by

$$\gamma \begin{pmatrix} \tau_{11} & \tau_{12} & z \\ \tau_{21} & \tau_{22} & b \\ z & b & \tau_{g-2} \end{pmatrix} = \begin{pmatrix} \tau_{22} & \tau_{21} & b \\ \tau_{12} & \tau_{11} & z \\ b & z & \tau_{g-2} \end{pmatrix}.$$

The non-empty intersection  $Y_{ij} = D_i \cap D_j$  of two boundary components is irreducible, since its toric part is a torus orbit. Geometrically, it is a  $\mathbb{P}^1$ -bundle over  $\mathcal{X}_{g-2}(\ell) \times_{\mathcal{A}_{g-2}(\ell)} \hat{\mathcal{X}}_{g-2}(\ell)$ . More precisely, it is the compactification of the pull back  $\sigma^*(\mathcal{P})$  under the Galois cover (this pullback is in fact the  $\ell$ -th power of Poincaré bundle on  $\mathcal{X}_{g-2}(\ell) \times_{\mathcal{A}_{g-2}(\ell)} \hat{\mathcal{X}}_{g-2}(\ell)$ , see [GL], but this does not matter for us).

**Lemma 5.3.** *We have*

- (i)  $\ell D_i|_{D_i \cap D_j} = -2\xi - 2f^*T_1 + f^*\mathcal{P}$
- (ii)  $\ell D_j|_{D_i \cap D_j} = -2\xi - 2f^*T_2 + f^*\mathcal{P}$

Here we denote a class on  $Y$  and its pullback to the Galois cover  $Y_{ij} = D_i \cap D_j$  by the same symbol. The same applies to the map  $f$  which is the analogue of the bundle projection in diagram (11).

*Proof.* To clarify notation we denote the trivialized theta divisor on  $D_i$  by  $\Theta^{(i)}$ . It follows from Proposition 4.5 that  $\Theta^{(i)}|_{D_i \cap D_j} = \xi + f^*T_1 - 1/2f^*\mathcal{P}$ . Here  $\xi = D_i \cap D_j \cap D_k$ . We thus obtain claim (i) from Proposition 5.1 i.e.

$$\ell D_i|_{D_i \cap D_j} = -2\Theta^{(i)}|_{D_i \cap D_j} = -2\xi - 2f^*T_1 + f^*\mathcal{P}.$$

Claim (ii) now follows from

$$\begin{aligned} \ell D_j|_{D_i \cap D_j} &= \gamma^*(\ell D_i)|_{D_i \cap D_j} \\ &= (\gamma|_{D_i \cap D_j})^*(\ell D_i|_{D_i \cap D_j}) = (\gamma|_{D_i \cap D_j})^*(-2\Theta^{(i)}|_{D_i \cap D_j}) \\ &= (\gamma|_{D_i \cap D_j})^*(-2\xi - 2f^*T_1 + f^*\mathcal{P}) = -2\xi - 2f^*T_2 + f^*\mathcal{P} \end{aligned}$$

where the last equality follows since  $\gamma$  interchanges  $z$  and  $b$ .  $\square$



The preimage of  $\Delta$  under the Galois cover is also reducible, more precisely  $\Delta(\ell) = \cup \Delta_{ijk}$  where  $\Delta_{ijk} = D_i \cap D_j \cap D_k$  with  $\Delta_{ijk} \cong \mathcal{X}_{g-2}(\ell) \times_{\mathcal{A}_{g-2}(\ell)} \hat{\mathcal{X}}_{g-2}(\ell)$ . We also note

**Lemma 5.4.** *We fix an identification  $D_i = \mathcal{X}_{g-1}(\ell)$  and thus also an identification  $\Delta_{ijk} = D_i \cap D_j \cap D_k = \mathcal{X}_{g-2}(\ell) \times_{\mathcal{A}_{g-2}(\ell)} \hat{\mathcal{X}}_{g-2}(\ell)$ . Then the following holds:*

$$(i) \quad D_i|_{D_i \cap D_j \cap D_k} = -2T_1 - \mathcal{P}$$

$$(ii) \quad D_j|_{D_i \cap D_j \cap D_k} = -2T_2 - \mathcal{P}$$

$$(iii) \quad D_k|_{D_i \cap D_j \cap D_k} = \mathcal{P}$$

where we again use the same symbol for a class on  $\Delta$  and its pullback under the Galois cover.

*Proof.* To see the first identity we have to restrict the equality  $D_i|_{D_i \cap D_j} = -2\xi - 2f^*T_1 + f^*\mathcal{P}$  to  $D_i \cap D_j \cap D_k = \xi$ . The claim then follows since  $\xi^2 = \xi f^*\mathcal{P}$ . The second identity follows in the same way from Lemma 5.3. The last claim follows since  $D_k|_{D_i \cap D_j \cap D_k} = \xi|\xi = \mathcal{P}$ .  $\square$

**Remark 5.5.** Note that (iii) is also consistent with the fact that  $N_{\Delta/Y} = N_{D_i \cap D_j \cap D_k / D_i \cap D_j} = D_k|_{D_i \cap D_j \cap D_k} = \mathcal{P}$ .

## 6 Combinatorics of the level cover

We will now need to deal more carefully with the combinatorics of the boundary of the level cover, to understand how the intersection numbers that we want to compute, which are of course well-defined on  $\mathcal{A}_g^{\text{Perf}}$  with no level, can first be computed by doing an honest computation on the variety  $\mathcal{A}_g^{\text{Perf}}(\ell)$ , and then how this computation on  $\mathcal{A}_g^{\text{Perf}}(\ell)$  can, due to its invariance under the deck transformation group of the cover, be reduced back to some computations without a level.

We denote by  $\sigma : \mathcal{A}_g^{\text{Perf}}(\ell) \rightarrow \mathcal{A}_g^{\text{Perf}}$  the level cover, which has degree

$$\nu_g(\ell) = |\text{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})| = \ell^{g(2g+1)}(1 - \ell^{-2}) \cdot \dots \cdot (1 - \ell^{-2g}). \quad (18)$$

Note that this is the degree of the map between the stacks. The degree of the maps between the varieties is  $|\text{PSP}(2g, \mathbb{Z}/\ell\mathbb{Z})|$  which is  $\nu_g(\ell)/2$ . For further use, we denote by  $d_g(\ell)$  the number of irreducible components  $D_i$

of the boundary  $\mathcal{A}_g^{\text{Perf}}(\ell)$ . This is equal to one half the number of points in  $(\mathbb{Z}/\ell\mathbb{Z})^{2g} \setminus \{0\}$ , which is

$$d_g(\ell) = \frac{1}{2}\ell^{2g}(1 - \ell^{-2g}). \quad (19)$$

The cover  $\sigma$  branches to order  $\ell$  along the boundary, cf. (17). Using Lemma 5.2 we get

$$\begin{aligned} a_N^{(g)} &= \langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\text{Perf}}} = \frac{1}{\nu_g(\ell)} \langle \sigma^* L^{G-N} \sigma^* D^N \rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} \\ &= \frac{\ell^N}{\nu_g(\ell)} \left\langle \sigma^* L^{G-N} \left[ \sum_i D_i^N + \sum_{i>j; a+b=N, a,b>0} \binom{N}{a} D_i^a D_j^b \right. \right. \\ &\quad \left. \left. + \sum_{i>j>k; a+b+c=N, a,b,c>0} \binom{N}{a, b, c} D_i^a D_j^b D_k^c \right] \right\rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} \end{aligned} \quad (20)$$

(where we remind the reader that for  $a + b + c = N$  by definition  $\binom{N}{a, b, c} = \frac{N!}{a!b!c!}$ ). We will now reduce this computation of the intersection numbers on a level cover to the intersection computation on the no level moduli space. Let us start with the first term.

**Lemma 6.1.** *The first term of equation (20) is given by*

$$\frac{\ell^N}{\nu_g(\ell)} \left\langle \sigma^* L^{G-N} \sum_i D_i^N \right\rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} = \frac{1}{2} \langle L^{G-N} (-2\Theta)^{N-1} \rangle_{\mathcal{X}_{g-1}^{\text{part}}}.$$

*Proof.* We denote by  $e_I(\ell)$  the degree of the map (as stacks) from each boundary component  $D_i$  of  $\mathcal{A}_g^{\text{Perf}}(\ell)$  to the (closure of the) universal family  $\mathcal{X}_{g-1}$  (this map is clearly onto). From the description of  $D_i$  as the universal level family  $\mathcal{X}_{g-1}(\ell)$ , mapping to  $\mathcal{X}_{g-1}$ , we get two factors for this degree — one is the degree of the map  $\mathcal{A}_{g-1}(\ell)$  to  $\mathcal{A}_{g-1}$  (thus it is  $\nu_{g-1}(\ell)$ ), and the other factor is the degree of the map of a single fiber of the family  $\mathcal{X}_{g-1}(\ell) \rightarrow \mathcal{A}_{g-1}(\ell)$  to a fiber of  $\mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$ , i.e of the map of an  $\ell$ -times principally polarized abelian variety of dimension  $g - 1$  to a principally polarized abelian variety of dimension  $g - 1$ . We thus have

$$e_I(\ell) = \nu_{g-1}(\ell) \ell^{2g-2}.$$

By Proposition 5.1 we obtain

$$\begin{aligned} \frac{\ell^N}{\nu_g(\ell)} \langle \sigma^* L^{G-N} \sum_i D_i^N \rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} &= \frac{\ell}{\nu_g(\ell)} \langle \sigma^* L^{G-N} \sum_i (-2\Theta^{(i)})^{N-1} D_i \rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} \\ &= \frac{\ell}{\nu_g(\ell)} d_g(\ell) e_I(\ell) \langle L^{G-N} (-2\Theta)^{N-1} \rangle_{\mathcal{X}_{g-1}^{\text{part}}}. \end{aligned}$$

To prove the lemma it remains thus to compute, substituting the expressions for  $d_g$  and  $\nu_g$  from formulae (19) and (18),

$$\begin{aligned} &\frac{\ell}{\nu_g(\ell)} d_g(\ell) e_I(\ell) \\ &= \frac{\frac{1}{2} \ell \ell^{2g} (1 - \ell^{-2g}) \frac{1}{2} \ell^{(g-1)(2g-1)} (1 - \ell^{-2}) \cdot \dots \cdot (1 - \ell^{-2g+2}) \ell^{2g-2}}{\frac{1}{2} \ell^{g(2g+1)} (1 - \ell^{-2}) \cdot \dots \cdot (1 - \ell^{-2g})} = \frac{1}{2}. \end{aligned}$$

□

**Remark 6.2.** Note that the factor  $1/2$  fully agrees with the proof of Proposition 2.2.

We have described the geometry of  $\beta_1 \setminus \beta_3$  in Section 3. If we work with the level cover  $\mathcal{A}_g^{\text{Perf}}(\ell)$ , then the irreducible components of  $\beta_1(\ell)$  are compactifications of the universal level families  $\mathcal{X}_{g-1}(\ell)$ , the components of the locus  $\Delta(\ell)$  are  $\mathcal{X}_{g-2}(\ell) \times_{\mathcal{A}_{g-2}(\ell)} \hat{\mathcal{X}}_{g-2}(\ell)$ , and  $\beta_2(\ell) \setminus \beta_3(\ell)$  is obtained by the gluing of  $\ell$  copies of the  $\ell$ 'th powers of the universal Poincaré bundle (see [GL] for a discussion of why this is  $\mathcal{P}^{\otimes \ell}$ , but this does not matter for our purposes here). With this description of the geometry, we can prove

**Lemma 6.3.** *The second term in formula (20) can be reduced to a computation on  $Y$  as follows: for fixed  $a, b > 0$  with  $a + b = N$*

$$\begin{aligned} &\frac{\ell^N}{\nu_g(\ell)} \left\langle \sigma^* L^{G-N} \sum_{i>j} D_i^a D_j^b \right\rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} \\ &= \frac{1}{8} \left\langle L^{G-N} (-2\xi - 2f^* T_1 + f^* \mathcal{P})^{a-1} (-2\xi - 2f^* T_2 + f^* \mathcal{P})^{b-1} \right\rangle_Y. \end{aligned}$$

*Proof.* We denote by  $d_{II}(\ell)$  the number of components of  $\bigcup_{i>j} (D_i \cap D_j)$  — note that all these components lie in  $\beta_2(\ell) \subset \mathcal{A}_g^{\text{Perf}}(\ell)$  and that each component of this set lies on only two boundary divisors, i.e. it is impossible to have  $D_i \cap D_j \subset D_k$  for  $i, j, k$  distinct. This follows since the second

Voronoi decomposition for  $g = 2$  is basic. For any fixed  $i$  we think of  $D_i$  as being a compactification  $\mathcal{X}_{g-1}^{\text{Perf}}(\ell)$  of  $\mathcal{X}_{g-1}(\ell)$ . The intersections  $D_i \cap D_j$ , being the closures of torus orbits in a toroidal compactification, are irreducible, and are the boundary components of  $\mathcal{X}_{g-1}^{\text{Perf}}(\ell)$ . The fibers of the map  $\mathcal{X}_{g-1}^{\text{part}}(\ell) \rightarrow \mathcal{A}_{g-1}^{\text{part}}(\ell)$  are corank 1 degenerations of abelian varieties and as such have  $\ell$  irreducible components. Hence the number of irreducible boundary components of  $\mathcal{X}_{g-1}^{\text{part}}(\ell)$  is equal to the number of irreducible boundary components of  $\mathcal{A}_{g-1}^{\text{part}}(\ell)$  multiplied by  $\ell$ . Thus

$$d_{II}(\ell) = \frac{\ell}{2} d_g(\ell) d_{g-1}(\ell)$$

where the factor  $1/2$  accounts for the fact that every intersection  $D_i \cap D_j$  is contained in exactly 2 boundary components. We also need to compute  $e_{II}(\ell)$ , the degree of the map from any  $D_i \cap D_j$  to  $\beta_2$ . Indeed, from our description of the geometry we see that the level cover over  $\beta_2 \setminus \beta_3$  is the composition of mapping  $\ell$  copies of the  $\mathbb{P}^1$ -bundle to one copy (so factor of  $\ell$  for the degree) together with mapping each component of  $\Delta(\ell)$  to  $\Delta$ , i.e. mapping  $\mathcal{X}_{g-2}(\ell) \times_{\mathcal{A}_{g-2}(\ell)} \hat{\mathcal{X}}_{g-2}(\ell) \rightarrow \mathcal{X}_{g-2} \times_{\mathcal{A}_{g-2}} \hat{\mathcal{X}}_{g-2}$ . The latter map has the degree equal to  $\nu_{g-2}(\ell)$ , which is the degree of the map of the base  $\mathcal{A}_{g-2}(\ell) \rightarrow \mathcal{A}_{g-2}$ , times the degree on the fibers, i.e. the square of the degree of the map of a  $\ell$ -times principally polarized abelian variety to a principally polarized abelian variety. Thus we get

$$e_{II}(\ell) = \ell \nu_{g-2}(\ell) \ell^{4(g-2)}.$$

Similarly to the first term above, to compare the intersection numbers on  $\mathcal{A}_g^{\text{Perf}}(\ell)$  and on  $\mathcal{A}_g^{\text{part}}(\ell)$  we need to compute the ratio

$$\begin{aligned} \frac{\ell^2 d_{(II)}(\ell) e_{(II)}(\ell)}{\nu_g(\ell)} &= \frac{\ell^2 \ell d_g(\ell) d_{g-1}(\ell) \ell \nu_{g-2}(\ell) \ell^{4(g-2)}}{2 \nu_g(\ell)} \\ &= \frac{\ell^{4+2g+2(g-1)+(g-2)(2g-3)+4(g-2)} (1 - \ell^{-2g})(1 - \ell^{-2g})(1 - \ell^{-2}) \cdots (1 - \ell^{4-2g})}{8 \ell^{g(2g+1)} (1 - \ell^{-2}) \cdots (1 - \ell^{-2g})} \\ &= \frac{1}{8}. \end{aligned}$$

This accounts for the factor  $1/8$  which occurs on the right hand side of the equation. The claim now follows from Lemma 5.3.  $\square$

The third term in (20) can also be dealt with in a similar fashion.

**Lemma 6.4.** *Assume that  $N < 3g - 3$  (in which case  $\beta_3$  can be ignored). Then for fixed  $a, b, c > 0$  with  $a + b + c = N$  we have*

$$\begin{aligned} & \frac{\ell^N}{\nu_g(\ell)} \langle \sigma^* L^{G-N} \sum_{i>j>k} D_i^a D_j^b D_k^c \rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)} \\ &= \frac{1}{12} \left\langle L^{G-N} (-2T_1 - \mathcal{P})^{a-1} (-2T_2 - \mathcal{P})^{b-1} \mathcal{P}^{c-1} \right\rangle_{\Delta}. \end{aligned}$$

*Proof.* The intersection of three boundary components  $D_i \cap D_j \cap D_k$  for different indices  $i, j$  and  $k$  can either be of local or of global type. In the latter case such an intersection lies entirely in  $\beta_3(\ell)$  and is irrelevant for us. Any intersection  $D_i \cap D_j \cap D_k$  of local type corresponds to a 2-dimensional cone in the perfect cone decomposition for  $g = 2$  and is thus a torus orbit and hence irreducible. Moreover the intersections  $D_i \cap D_j \cap D_k$  of local type are exactly the boundary components of  $\Delta(\ell)$  and no component of  $\Delta(\ell)$  is contained in 4 boundary components by Lemma 5.2. Similarly to the above case, we denote by  $d_{III}(\ell)$  the number of irreducible components of  $\bigcup_{i>j>k} D_i \cap D_j \cap D_k$ , where the union is only taken over the intersections of local type. This is equal to the number of irreducible components of  $\Delta(\ell)$ . For fixed  $i$ , the number of components of  $\Delta(\ell)$  contained in  $D_i$  is equal to the number of boundary components of  $\mathcal{X}_{g-1}^{\text{Perf}}(\ell)$  times  $\ell$  — this is as in the second term computation above, since the components of  $\Delta(\ell)$  correspond exactly to the intersection of two boundary components of  $\mathcal{X}_{g-1}^{\text{Perf}}(\ell)$  and any such component contains 2 components of  $\Delta(\ell)$ . We thus have

$$d_{III}(\ell) = \frac{\ell}{3} d_g(\ell) d_{g-1}(\ell)$$

where the denominator 3 comes from the fact that every component of  $\Delta(\ell)$  lies in exactly 3 boundary components. We also need to compute  $e_{III}(\ell)$ , the degree of the map of each component of  $\Delta(\ell)$  to  $\Delta$ . The degree  $e_{II}(\ell)$  that we computed above was for mapping  $\ell$  copies of a  $\mathbb{P}^1$ -bundle to one copy, together with some map of the base. Now we have the same map of the base, and are mapping just one point taken from the union of  $\ell$  copies of  $\mathbb{P}^1$  in the fiber to one point, and thus  $e_{III}(\ell) = e_{II}(\ell)/\ell$ . We compute

$$\frac{\ell^3 d_{III}(\ell) e_{III}(\ell)}{\nu_g(\ell)} = \frac{\ell^3 \ell d_g(\ell) d_{g-1}(\ell) \nu_{g-2}(\ell) \ell^{4(g-2)}}{3 \nu_g(\ell)} = \frac{1}{12}$$

since this is the same computation as we had for the second term, but with 3 instead of 2 in the denominator. The claim now follows from Lemma 5.4.  $\square$

Combining these results, we finally get the following

**Proposition 6.5.** *For  $2g - 1 \leq N < 3g - 3$  the intersection numbers are*

$$\begin{aligned}
a_N^{(g)} &= \langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\text{Perf}}} = \frac{1}{2} \langle L^{G-N} (-2\Theta)^{N-1} \rangle_{\mathcal{X}_{g-1}^{\text{part}}} \\
&+ \frac{1}{8} \sum_{a+b=N, a, b > 0} \binom{N}{a} \langle L^{G-N} (-2\xi - 2f^*T_2 + f^*\mathcal{P})^{a-1} (-2\xi - 2f^*T_1 + f^*\mathcal{P})^{b-1} \rangle_Y \\
&+ \frac{1}{12} \sum_{a+b+c=N, a, b, c > 0} \binom{N}{a, b, c} \langle L^{G-N} (-2T_2 - \mathcal{P})^{a-1} (-2T_1 - \mathcal{P})^{b-1} \mathcal{P}^{c-1} \rangle_{\Delta} \\
&=: (I) + (II) + (III)
\end{aligned}$$

where we have numbered the terms for future reference.

**Remark 6.6.** The factors  $1/2$ ,  $1/8$  and  $1/12$  are due to the stackiness of  $\mathcal{A}_g^{\text{Perf}}$  and the toroidal geometry. The first factor comes from the fact that the map of stacks  $\mathcal{X}_{g-1} \rightarrow D$  has degree 2. Similarly, the factor  $1/8$  comes from the fact that each component of  $Y(l)$  lies on two boundary components (accounting for one factor 2) and each of these boundary components accounts for a further factor 2. Finally, each components of  $\Delta(\ell)$  is the intersection of 3 boundary components of  $\mathcal{A}_g^{\text{Perf}}(\ell)$ . From this one would naively expect a factor  $1/3 \cdot 2^3 = 1/24$ . Note, however, that  $\mathcal{X}_{g-1}$  and  $Y$  are families of abelian varieties, resp. degenerate abelian varieties where the fibers have an involution, whereas the situation is different for  $\Delta$ , which is actually a substack of  $\mathcal{A}_g^{\text{Perf}}$  unlike  $\mathcal{X}_{g-1}$  or  $Y$ . Hence we obtain the factor  $1/12$  rather than  $1/24$ . However, from these considerations it is not immediately clear to us how to see that we have described the entire stabilizers of these loci, and thus we gave the rigorous proofs above using the combinatorics of level covers.

Thus we have expressed the intersection number on  $\mathcal{A}_g^{\text{Perf}}$  as a sum of 3 intersection numbers on different loci (without level cover!). Each of the three summands is the intersection of classes for which we have an explicit geometric description. In the following sections we will compute these three intersection numbers.

## 7 Intersection numbers on $\Delta$ (term (III))

We are now ready to compute the intersection numbers appearing in formula (20) where we will start with term (III). This requires the computation of

certain intersection numbers on  $\Delta$ , and we will now determine completely the intersection theory of divisors on  $\Delta$ . We will in fact compute all the intersection numbers

$$\langle L^{G-N} T_1^l T_2^m \mathcal{P}^n \rangle_\Delta = \langle L^{G-N} h_*(T_1^l T_2^m \mathcal{P}^n) \rangle_{\mathcal{A}_{g-2}}$$

where the equality follows from the projection formula [Fu, Proposition 8.3 (c)]. In the following sections we will reduce the computations of the two other terms to working on  $\Delta$ .

**Theorem 7.1.** *The pushforward  $h_*(T_1^l T_2^m \mathcal{P}^n) = 0$  unless  $l = m$  and  $l + m + n = 2g - 4$ , in which case it is*

$$h_*(T_1^{g-k-2} T_2^{g-k-2} \mathcal{P}^{2k}) = (-1)^k \frac{(g-2)!(2k)!(g-2-k)!}{k!} [\mathcal{A}_{g-2}]. \quad (21)$$

We start by computing the pushforwards of the intersections of the divisors  $T_1$  and  $T_2$ .

**Lemma 7.2.** *The pushforward  $h_*(T_1^l T_2^m)$  is zero unless  $l = m = g - 2$ , in which case it is equal to  $((g-2)!)^2 [\mathcal{A}_{g-2}]$ .*

*Proof.* The projection map  $h$  in diagram (11) decomposes as a composition of  $p_1$  and  $pr_2$ , resp. of  $p_2$  and  $pr_1$ ; moreover, the divisors  $T_1$  and  $T_2$  are the pullbacks of the universal  $(g-2)$ -dimensional theta divisor  $\Theta_{g-2} \subset \mathcal{X}_{g-2}$  (symmetric and trivialized along the zero section, as always) under  $pr_1$  and  $pr_2$  respectively. Thus

$$h_*(T_1^l T_2^m) = (p_1 \circ pr_2)_*(pr_1^* \Theta_{g-2}^l \cdot pr_2^* \Theta_{g-2}^m) = p_{1*}(pr_{2*}(pr_1^* \Theta_{g-2}^l \cdot \Theta_{g-2}^m))$$

again by the projection formula. Since flat pullback commutes with proper pushforward [Fu, Proposition 1.7] we have

$$pr_{2*} \circ pr_1^* = p_1^* \circ p_{2*}.$$

Hence we can rewrite the pushforward above as

$$p_{1*}(pr_{2*}(pr_1^* \Theta_{g-2}^l \cdot \Theta_{g-2}^m)) = p_{1*}(p_1^*(p_{2*} \Theta_{g-2}^l \cdot \Theta_{g-2}^m)).$$

The advantage of doing this is that the pushforward under  $p_{2*}$ , is simply the pushforward of the universal symmetric theta divisor trivialized along the zero section on the universal family  $\mathcal{X}_{g-2} \rightarrow \mathcal{A}_{g-2}$ , which we know by (9) to be zero for  $l > g - 2$ . On the other hand  $p_{2*} \Theta_{g-2}^l = 0$  for  $l < g - 2$  for

dimension reasons. Since this is symmetric in  $l$  and  $m$  the only remaining possibility is  $l = m = g - 2$  in which case we get

$$\begin{aligned} p_{1*}(p_1^*(p_{2*}\Theta_{g-2}^{g-2}) \cdot \Theta_{g-2}^{g-2}) &= (g-2)!p_{1*}(p_1^*[\mathcal{A}_{g-2}]) \cdot \Theta_{g-2}^{g-2} \\ &= (g-2)!p_{1*}(\Theta_{g-2}^{g-2}) = ((g-2)!)^2[\mathcal{A}_{g-2}]. \end{aligned}$$

□

One could now proceed to compute directly the pushforwards of intersections of  $T_1$  and  $T_2$  with  $\mathcal{P}$ , but there is an easier way. We will use the automorphism group of  $\Delta$ . In Section 4 we have already introduced the shift operator  $s : \Delta \rightarrow \Delta$  defined by  $s(z, b) = (z + b, b)$ . Let

$$V_N := (s^N)^*(T_1) = T_1 + N^2T_2 + N\mathcal{P}$$

where the last equality follows from repeated application of Lemma 4.2.

**Lemma 7.3.** *For any  $N, l, m$  the pushforward  $h_*(V_N^l T_2^m) = h_*(T_1^l T_2^m)$ .*

*Proof.* By Lemma 4.2 we have  $V_N^l T_2^m = (s^N)^*(T_1^l T_2^m)$ , and the result then follows since  $s^N$  is an automorphism of  $\Delta$  satisfying  $h \circ s^N = h$ . □

We can now use this to complete the computation of all pushforwards on  $\Delta$ .

*Proof of theorem 7.1.* We will perform induction in  $n$  — the power of the Poincaré bundle in  $h_*(T_1^l T_2^m \mathcal{P}^n)$ . First note that for  $n = 0$  we know the result to be true — this is the content of Lemma 7.2. Now assume the result to be true if the power of the Poincaré bundle is strictly less than  $n$ , and let us prove the result for  $n$  and arbitrary powers  $l$  and  $m$ . Using Lemma 7.3, we see that for all  $A, B, N$

$$h_*((T_1 + N^2T_2 + N\mathcal{P})^A T_2^B) = \delta_{A, g-2} \delta_{B, g-2} ((g-2)!)^2$$

where  $\delta$  denotes Kronecker's delta. The pushforward on the left-hand side is a polynomial in  $N$ , which thus has to be constant. In particular, if we extract the  $N^n$  term, it must be identically zero, i.e. for all  $n > 0$  we have

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{A}{i, n-2i, A-n+i} h_*(T_1^{A-n+i} T_2^{B+i} \mathcal{P}^{n-2i}) = 0. \quad (22)$$

Observe that this pushforward contains one term which we would like to compute, namely that where  $\mathcal{P}^n$  appears, while all the other terms have



lower powers of  $\mathcal{P}$ . In particular we know inductively that unless  $(A - n + i) + (B + i) + (n - 2i) = A + B = 2g - 4$ , and  $A - n + i = B + i$  for some (and thus for all)  $i$ , all the terms with lower powers of  $\mathcal{P}$  are zero, and thus (of course the binomial coefficient is non-zero) the term with  $h_*(T_1^{A-n} T_2^B \mathcal{P}^n)$  is also zero unless  $A + B = 2g - 4$  and  $A - n = B$ . Thus we have inductively proven that the zeroes claimed for terms with  $\mathcal{P}^n$  are there, and it remains to obtain the formula for the only non-zero pushforward with  $\mathcal{P}^n$  (for the case of  $n$  even — for  $n$  odd still all the pushforwards are zero).

In this case we have  $A - n = B$  and  $2B = 2g - 4 - n$ , so denoting  $n = 2k$  we get  $B = g - 2 - k$ , and  $A = g - 2 + k$ . Since the above equation determines  $h_*(T_1^{g-2-k} T_2^{g-2-k} \mathcal{P}^{2k})$  uniquely, to prove that this pushforward is equal to what we want, it is enough to verify that plugging in the claimed formula for it yields zero for the sum above. Indeed, by plugging in the values of the pushforwards in (22) above we get

$$\begin{aligned} & \sum_{i=0}^k \binom{g-2+k}{i, 2k-2i, g-2-k+i} (-1)^{k-i} \frac{(g-2)!(2k-2i)!(g-2-k+i)!}{(k-i)!} \\ &= (g-2)!(g-2+k)! \sum_{i=0}^k (-1)^{k-i} \frac{(2k-2i)!(g-2-k+i)!}{i!(2k-2i)!(g-2-k+i)!(k-i)!} \\ &= (g-2)!(g-2+k)! \sum_{i=0}^k (-1)^{k-i} \frac{1}{i!(k-i)!} = 0 \end{aligned}$$

where we have used the standard identity

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} = 0$$

for the last equality.  $\square$

In order to compute term (III) we must compute intersection numbers of the form

$$\begin{aligned} & \langle L^{G-N} (-2T_2 - \mathcal{P})^{a-1} (-2T_1 - \mathcal{P})^{b-1} \mathcal{P}^{c-1} \rangle_{\Delta} \\ &= \langle L^{G-N} h_*((-2T_2 - \mathcal{P})^{a-1} (-2T_1 - \mathcal{P})^{b-1} \mathcal{P}^{c-1}) \rangle_{\mathcal{A}_{g-2}} \end{aligned}$$

with  $a + b + c = N$ . For  $a + b \leq 2g - 1$  we set

$$C_g^{a,b} := (-1)^{a+b+g} (g-2)! \sum_{i=0}^{\min(a-1, b-1)} \frac{(-4)^i (a-1)!(b-1)!(2g-4-2i)!}{i!(a-1-i)!(b-1-i)!(g-2-i)!}$$

and  $C_g^{a,b} := 0$  otherwise.

This sum, as a generalized hypergeometric sum, can be evaluated explicitly, using Maple [Ma], which implements the method described for example in [GKP], to yield the following answer

$$C_g^{a,b} = (-1)^{a+b+g} (2g-4)! \frac{\Gamma(\frac{5}{2}-g) \Gamma(\frac{1}{2}+a+b-g)}{\Gamma(\frac{3}{2}+a-g) \Gamma(\frac{3}{2}+b-g)} \quad (23)$$

where by using the standard properties of the  $\Gamma$  function one can further substitute

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \quad \text{and} \quad \Gamma\left(\frac{1}{2}-n\right) = \frac{(-2)^n}{(2n-1)!!} \sqrt{\pi} \quad (24)$$

for  $n \in \mathbb{Z}_{\geq 0}$ .

With this notation we have the following

**Corollary 7.4.** *The pushforward*

$$h_*((-2T_2 - \mathcal{P})^{a-1}(-2T_1 - \mathcal{P})^{b-1}\mathcal{P}^{c-1}) = 0$$

unless  $a+b+c = 2g-1$ , in which case it is

$$h_*((-2T_2 - \mathcal{P})^{a-1}(-2T_1 - \mathcal{P})^{b-1}\mathcal{P}^{c-1}) = C_g^{a,b}[\mathcal{A}_{g-2}].$$

*Proof.* All the monomials appearing in these pushforwards are zero unless  $(a-1) + (b-1) + (c-1) = 2g-4$ , so the vanishing is easy, and we are left to compute for  $a+b+c = 2g-1$ , using formula (21)

$$\begin{aligned} & h_*((-2T_2 - \mathcal{P})^{a-1}(-2T_1 - \mathcal{P})^{b-1}\mathcal{P}^{c-1}) \\ &= (-1)^{a+b-2} \sum_{i=0}^{\min(a-1, b-1)} \binom{a-1}{i} \binom{b-1}{i} h_*((2T_2)^i (2T_1)^i \mathcal{P}^{2g-4-2i}) \\ &= (-1)^{a+b} \sum_{i=0}^{\min(a-1, b-1)} \frac{(-1)^{g-i} 4^i (a-1)! (b-1)! (g-2)! (2g-4-2i)!}{i! (a-1-i)! (b-1-i)! (g-2-i)!}. \end{aligned}$$

□

Combining this with the previous results we finally obtain

**Theorem 7.5.** *The term (III) in (20) is equal to zero in all cases except  $N = 2g - 1$  when it is*

$$(III) = \frac{(-1)^{g+1}(4g^2 - 8g + 7)(2g - 4)!}{12(2g - 1)} + \frac{16^g(g - 2)!(g - 1)!}{192(2g - 1)} \\ + \frac{(2g - 1)!(2g - 4)!}{12} \sum_{a=1}^{2g-3} \sum_{k=1}^{2g-2-a} \frac{(-1)^{g+a+k+1} \left[\frac{3}{2} - g + a\right]_k}{(k + 1)(2g - a - k - 2)!a! \left[\frac{5}{2} - g\right]_k}$$

where  $[z]_k := z \cdot (z + 1) \cdot \dots \cdot (z + k - 1)$  denotes the so-called Pochhammer symbol.

*Proof.* This is a straightforward calculation. Indeed, by the above corollary all the pushforwards in

$$(III) = \frac{1}{12} \sum_{a+b+c=N, a,b,c>0} \binom{N}{a, b, c} \\ \times \left\langle L^{G-N} h_* \left( (-2T_2 - \mathcal{P})^{a-1} (-2T_1 - \mathcal{P})^{b-1} \mathcal{P}^{c-1} \right) \right\rangle_{\mathcal{A}_{g-2}}$$

are zero unless  $N = a + b + c = 2g - 1$ , in which case we get

$$(III) = \frac{1}{12} \langle L^{(g-2)(g-1)/2} \rangle_{\mathcal{A}_{g-2}} \sum_{a+b+c=2g-1, a,b,c>0} \frac{(2g - 1)!}{a!b!c!} C_g^{a,b}.$$

Using Maple to sum this (and simplifying by hand each time in between summations) yields the claimed formula.  $\square$

**Remark 7.6.** The resulting expression can also be rewritten in terms of the hypergeometric functions  ${}_3F_2$ .

## 8 Intersection numbers on $Y$ (term (II))

For term (II) in (20) we must compute intersection numbers of the form

$$\langle L^{G-N} (-2\xi - 2f^*T_1 + f^*\mathcal{P})^a (-2\xi - 2f^*T_2 + f^*\mathcal{P})^b \rangle_Y.$$

We will compute the intersection number on  $Y$  by pushing the computation down to  $\Delta$  and then to  $\mathcal{A}_{g-2}$ , using the results of the previous section. Recall from diagram (11) the composite map  $\pi = h \circ f : Y \rightarrow \mathcal{A}_{g-2}$ . We will need to compute the pushforwards under this map of various products of divisors. In doing so, we shall distinguish between the cases when  $\xi$  is one of the factors or is not. Note that by (13) we can always assume the power of  $\xi$  to be at most one. We first make the following observation

**Lemma 8.1.** *For any class  $x \in \mathrm{CH}^*(\Delta)$  the following identities hold:*

$$\pi_* f^* x = h_*(f_* f^* x) = 0 \quad \text{and} \quad \pi_*(\xi f^* x) = h_*(x) \in \mathrm{CH}^*(\mathcal{A}_{g-2}).$$

*Proof.* Note that the fiber dimension of the map  $f$  is 1. Hence the first claim follows for dimension reasons. The second claim follows since  $\xi$  is a section of  $f$ .  $\square$

**Proposition 8.2.** *The following equality holds:*

$$\begin{aligned} \pi_*((-2\xi - 2f^*T_1 + f^*\mathcal{P})^{a-1}(-2\xi - 2f^*T_2 + f^*\mathcal{P})^{b-1}) \\ = \begin{cases} 2C_g^{a,2g-1-a}[\mathcal{A}_{g-2}] & \text{if } a+b = 2g-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

*Proof.* We compute

$$\begin{aligned} & (-1)^{a+b} \pi_* \left( (-2\xi - 2f^*T_1 + f^*\mathcal{P})^{a-1} (-2\xi - 2f^*T_2 + f^*\mathcal{P})^{b-1} \right) \\ &= \pi_* \left( \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \binom{a-1}{i} \binom{b-1}{j} (2f^*T_1)^i (2f^*T_2)^j (2\xi - f^*\mathcal{P})^{a+b-2-i-j} \right). \end{aligned}$$

Denoting  $A := a + b - 2 - i - j$  and using (13) repeatedly we obtain

$$\begin{aligned} (2\xi - f^*\mathcal{P})^A &= (\xi + (\xi - f^*\mathcal{P}))^A = \xi^A + (\xi - f^*\mathcal{P})^A \\ &= \xi^A + (\xi - f^*\mathcal{P})^{A-1}(\xi - f^*\mathcal{P}) = \xi^A + (\xi - f^*\mathcal{P})^{A-1}(-f^*\mathcal{P}) = \dots \\ &= \xi^A + (\xi - f^*\mathcal{P})(-f^*\mathcal{P})^{A-1} = (1 + (-1)^{A-1})\xi(f^*\mathcal{P})^{A-1} - (f^*\mathcal{P})^A. \end{aligned}$$

We now apply Lemma 8.1 to get

$$\begin{aligned} & \pi_*((2f^*T_1)^i (2f^*T_2)^j (2\xi - f^*\mathcal{P})^A) \\ &= \pi_* \left( ((1 + (-1)^{A-1})\xi(f^*\mathcal{P})^{A-1} - (f^*\mathcal{P})^A) (2f^*T_1)^i (2f^*T_2)^j \right) \\ &= h_*((1 + (-1)^{A-1})\mathcal{P}^{A-1}(2T_1)^i (2T_2)^j). \end{aligned}$$

By Theorem (7.1) this pushforward is non-zero only if  $i = j$  and  $a + b = 2g - 1$ , in which case it is equal to

$$\begin{aligned} & h_*((1 + (-1)^{2g-4-2i})\mathcal{P}^{2g-4-2i}(2T_1)^i (2T_2)^i) \\ &= 2(-1)^{g-2-i} 4^i \frac{(g-2)!(2g-4-2i)!i!}{(g-2-i)!} [\mathcal{A}_{g-2}]. \end{aligned}$$

Substituting this into the sum above gives the desired result.  $\square$

We can now finish the computation of the second term for the intersection number — this is just a matter of combining the known results, and computing carefully.

**Theorem 8.3.** *Term (II) in (20) is equal to*

$$(II) = \begin{cases} \frac{-a_0^{(g-2)}}{64(2g-1)} (2^{4g}(g-1)!(g-2)! + 32(-1)^g(2g-3)!) & \text{if } N = 2g-1 \\ 0 & \text{otherwise.} \end{cases}$$

where the explicit value of  $a_0^{(g-2)}$  can of course be substituted from formula (1).

*Proof.* By definition, (II) is a sum involving the terms  $\pi_*((-2\xi - 2f^*T_2 + f^*\mathcal{P})^{a-1}(-2\xi - 2f^*T_1 + f^*\mathcal{P})^{b-1})$ . Since these are all zero unless  $a+b = 2g-1$  we obtain the vanishing of term (II) as claimed. It follows from the definition of term (II) and Proposition 8.2 that for  $N = 2g-1$

$$(II) = \frac{1}{8} \langle L^{(g-2)(g-1)/2} \rangle_{\mathcal{A}_{g-2}} \sum_{a+b=2g-1, a,b>0} \frac{(2g-1)!}{a!b!} \cdot 2C_g^{a,b}.$$

Note now that  $C_g^{a,b}$  is symmetric in  $a$  and  $b$ . Thus it is enough to compute the sum for  $a \leq b = 2g-1-a$ , i.e. we have

$$(II) = \frac{1}{2} \sum_{a=1}^{g-1} \binom{2g-1}{a} C_g^{a, 2g-1-a} a_0^{(g-2)}.$$

We now use Maple to sum the terms given explicitly by formula (23), using the explicit factorial expressions for  $\Gamma$ -functions (for each  $\Gamma$ -function factor the sign of  $n$  in (24) is the same for all terms in the sum). This summation yields the result as claimed.  $\square$

## 9 Intersection numbers on $\mathcal{X}_{g-1}^{\text{part}}$ (term I)

In this section we finish the computation of the number  $a_{2g-1}^{(g)}$ , together with the proof that the next  $g-2$  numbers after it are zero, by computing term (I) in formula (20). As in the previous section, we eventually reduce this computation to a computation on  $\Delta$  and thus get the resulting number. The crucial ingredient in this reduction is a use of the Grothendieck-Riemann-Roch theorem. Recalling that in the range  $N < 3g-3$  which we are interested in,  $L^{G-N}$  is zero on  $\beta_3$ , and that  $L$  is a pullback from the Satake

compactification, we get from Lemma 6.1

$$(I) = \frac{1}{2}(-2)^{N-1} \langle L^{G-N} \Theta^{N-1} \rangle_{\mathcal{X}_{g-1}^{\text{part}}} = \frac{1}{2}(-2)^{N-1} \langle L^{G-N} \pi_*(\Theta^{N-1}) \rangle_{\mathcal{A}_{g-1}^{\text{part}}} \quad (25)$$

where  $\pi : \mathcal{X}_{g-1}^{\text{part}} \rightarrow \mathcal{A}_{g-1}^{\text{part}}$  is the partial compactification of the universal family in genus  $g-1$ .

Thus we need to compute the pushforward of  $\Theta$  on the partial compactification of the universal family — note that formula (9) only applies on the open part  $\mathcal{X}_{g-1} \rightarrow \mathcal{A}_{g-1}$ . Unfortunately we do not know a way to use a numerical argument similar to the one used in [FC] and explained in [vdG2] to compute the pushforward on  $\mathcal{A}_{g-1}^{\text{part}}$  directly, and thus will use the following identity

$$\pi_*(e^{\Theta} F) = e^{D/8} \quad (26)$$

where  $D$  denotes the class of the boundary of  $\mathcal{A}_{g-1}^{\text{part}}$  and  $F := \text{Td}^{\vee}(\mathcal{O}_{\Delta})^{-1}$ . Recall that  $\Delta$  is the class on the stack  $\mathcal{X}_{g-1}^{\text{part}}$  which for all levels  $n \geq 3$  is given by the union of the singular loci of the fibers of the extended universal family  $\pi : \mathcal{X}_{g-1}^{\text{part}}(n) \rightarrow \mathcal{A}_{g-1}^{\text{part}}(n)$ . Formula (26) was proven in [vdG2, Theorem 4.9] in cohomology, but using [EV, Theorem 5.2] it follows that it also holds in the Chow ring. This identity allows us to express term (I) in terms of the pushforwards that we already know.

To simplify the notation, we denote the coefficients of the power series expansion of the Todd class

$$\sum_{i=0}^{\infty} b_i x^i := \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}$$

where  $B_k$  are Bernoulli numbers, i.e. we set  $b_0 = 1, b_1 = \frac{1}{2}$  and for  $k \geq 1$  let  $b_{2k+1} = 0$ , and  $b_{2k} = (-1)^{k-1} \frac{B_k}{(2k)!}$ .

**Lemma 9.1.** *Let  $Z = E \cap F$  be a complete intersection of codimension 2 on a smooth manifold  $X$ . Then*

$$\text{Td}^{\vee}((\mathcal{O}_Z)^{-1}) = \sum_{i,j,k \geq 0} \sum_{a=0}^k \frac{(-1)^k b_i b_j}{(k+1)a!(k-a)!} E^{i+a} F^{j+k-a}. \quad (27)$$

*Proof.* We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-E - F) \longrightarrow \mathcal{O}_X(-E) \oplus \mathcal{O}_X(-F) \longrightarrow \mathcal{I}_Z \longrightarrow 0.$$

It then follows from the multiplicativity of the Todd class that

$$\mathrm{Td}((\mathcal{O}_Z)^{-1}) = \mathrm{Td}(\mathcal{I}_Z) = \left( \frac{-E}{1-e^E} \right) \left( \frac{-F}{1-e^F} \right) \left( \frac{-E-F}{1-e^{E+F}} \right)^{-1}$$

and thus (note that taking the dual means changing the signs of  $E$  and  $F$ )

$$\mathrm{Td}^\vee((\mathcal{O}_Z)^{-1}) = \left( \frac{E}{1-e^{-E}} \right) \left( \frac{F}{1-e^{-F}} \right) \left( \frac{1-e^{-E-F}}{E+F} \right). \quad (28)$$

The claim then follows by expanding the power series.  $\square$

**Remark 9.2.** Notice that this formula is written in terms of  $E$  and  $F$ ; however, in each degree it is a symmetric polynomial of  $E$  and  $F$ , and as such is expressible in terms of the elementary symmetric polynomials  $Z = EF$  and  $c_1(N_{Z/X}) = E + F$ . It is in this sense that we understand this formula.

Let us further denote by

$$b_{n,m} := \sum_{i=0}^n \sum_{j=0}^m \frac{(-1)^{n+m-i-j} b_i b_j}{(n+m-i-j+1)(n-i)!(m-j)!}$$

the coefficient of  $E^n F^m$  in (27). We were surprised to find out that these coefficients admit a remarkably simple formula, which we have been unable to find in the literature.

**Lemma 9.3.** *The coefficients above can be computed as follows:*

$$b_{n,m} = \begin{cases} (-1)^{\frac{n-m}{2}} \frac{B_{\frac{n+m}{2}}}{(n+m)!} & \text{if } n+m \text{ is even and } nm \neq 0 \\ 1 & \text{if } n = m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We start by noting the following straightforward identity:

$$\frac{1-e^{-E-F}}{(1-e^{-E})(1-e^{-F})} = \frac{1}{1-e^{-E}} + \frac{1}{1-e^{-F}} - 1.$$

Multiplying this by  $EF$ , we recognize the expressions for the Todd classes of  $E$  and  $F$  (multiplied by  $F$  and  $E$ , respectively) on the right. Finally we divide by  $E+F$ , getting expression (28) on the left, and use

$$\frac{E^{2k} + F^{2k}}{E+F} = E^{2k-1}F - E^{2k-2}F^2 + \dots - E^2F^{2k-2} + EF^{2k-1}$$

on the right (note that there are no linear terms here, unlike the Todd class).  $\square$

We now want to apply this to  $\Delta$  considered as a codimension 2 substack of  $\mathcal{X}_{g-1}^{\text{part}}$ . This is a complete intersection in a stack sense: if we go to level covers then every component of  $\Delta(\ell)$  in  $\mathcal{X}_{g-1}^{\text{part}}(\ell)$ , which we can identify with a fixed boundary component  $D_i$ , is a complete intersection of two further boundary components  $D_j$  and  $D_k$ . By Proposition 4.3 we have

$$N_{\Delta/\mathcal{X}_{g-1}^{\text{part}}} = \mathcal{P} \oplus (\mathcal{P}^{-1} \otimes T_2^{-2})$$

and thus we can use  $E = \mathcal{P}$  and  $F = -2T_2 - \mathcal{P}$  in the corollary above.

**Proposition 9.4.** *The first term in (6.5) is zero unless  $N = 2g - 1$ , in which case it is equal to*

$$(I) = \frac{(2g-2)!}{2^g(g-1)!} a_{g-1}^{(g-1)} + (-1)^g (2g-3)! a_0^{(g-2)} \sum_{m=1}^{g-1} \frac{(-1)^m 2^{2m+2-2g} B_m}{(2g-2m-1)!(2m)!}$$

*Proof.* Expanding identity (26) in power series and extracting the degree  $N-1$  term on the left, we get

$$\begin{aligned} \frac{\pi_*(\Theta^{N-1})}{(N-1)!} + \sum_{k=1}^{N-1} \pi_* \left( \frac{\Theta^{N-1-k}}{(N-1-k)!} \Delta \sum_{n=1}^{k-1} b_{n,k-n} \mathcal{P}^{n-1} (-2T_2 - \mathcal{P})^{k-n-1} \right) \\ = \frac{D^{N-g}}{8^{N-g}(N-g)!}. \end{aligned}$$

Computing an intersection number of any divisor class with  $\Delta$  is the same as restricting to the locus  $\Delta \subset \mathcal{A}_g^{\text{Perf}}$ . From Corollary 4.6 we know that  $\Theta|_{\Delta} = T_1 + \frac{1}{2}\mathcal{P}$ . The map  $\pi$  restricted to  $\Delta$  is the map  $pr_1 : \Delta \rightarrow \mathcal{X}_{g-2} = \mathcal{A}_{g-1}^{\text{part}} \setminus \mathcal{A}_{g-1}$ , and  $L$  on  $\Delta$  is a pullback from  $\mathcal{A}_{g-2}$ , so we are in the setup of the previous sections, and know all the pushforwards from Theorem (7.1). We can thus compute

$$\begin{aligned} (I) &= (-1)^{N-1} 2^{N-2} \langle L^{G-N} \pi_*(\Theta^{N-1}) \rangle_{\mathcal{A}_{g-1}^{\text{part}}} \\ &= (-1)^{N-1} 2^{N-2} (N-1)! \left[ \left\langle L^{G-N} \frac{D^{N-g}}{8^{N-g}(N-g)!} \right\rangle_{\mathcal{A}_{g-1}^{\text{part}}} \right. \\ &\quad \left. - \sum_{k=1}^{N-1} \left\langle \frac{L^{G-N}}{2} \sum_{n=1}^{k-1} \frac{b_{n,k-n}}{(N-1-k)!} \right. \right. \\ &\quad \left. \left. \times h_* \left( (T_1 + \mathcal{P}/2)^{N-1-k} \mathcal{P}^{n-1} (-2T_2 - \mathcal{P})^{k-n-1} \right) \right\rangle_{\mathcal{A}_{g-2}} \right]. \end{aligned}$$



Here the factor  $1/2$  in the expression  $L^{G-N}/2$  comes from the fact that we consider  $\Delta$  as a stack, i.e. we have an extra involution which acts trivially on the underlying variety (cf. also the proof of [vdG2, Lemma 4.8]).

Since all the pushforwards on the right are zero unless  $N = 2g - 1$ , and so is the first term on the right, it follows that (I) is zero unless  $N = 2g - 1$ . In this case we deduce from Corollary 7.4

$$h_* \left( (T_1 + \mathcal{P}/2)^{2g-2-k} \mathcal{P}^{n-1} (-2T_2 - \mathcal{P})^{k-n-1} \right) = \frac{(-1)^{-k-1}}{2^{2g-2-k}} C_g^{k-n, 2g-k-1}$$

and thus get in that case

$$(I) = 2^{2g-3} (2g-2)! \left( \frac{a_{g-1}^{(g-1)}}{8^{g-1} (g-1)!} - \frac{1}{2} a_0^{(g-2)} \sum_{k=1}^{2g-2} \sum_{n=1}^{k-1} \frac{(-1)^k b_{n,k-n}}{2^{2g-2-k} (2g-2-k)!} C_g^{k-n, 2g-k-1} \right).$$

We can now use the explicit expression for the coefficients  $b_{n,m}$  obtained in Lemma 9.3. It follows that the coefficient  $b_{n,k-n}$  in the formula above is zero unless  $k = 2m$ , in which case  $b_{n,2m-n} = (-1)^{m+n} \frac{B_m}{(2m)!}$ , so that the double sum above simplifies to

$$\sum_{m=1}^{g-1} \frac{(-1)^m B_m}{2^{2g-2-2m} (2g-2-2m)! (2m)!} \sum_{n=1}^{2m-1} (-1)^n C_g^{2m-n, 2g-2m-1}.$$

We then use Maple to compute the sum over  $n$  explicitly, getting a simple expression  $-\frac{(2g-3)!}{2^{g-1-2m}}$  for it (this is again an application of the algorithm for computing sums with hypergeometric terms described in [GKP]), and end up with the expression as claimed.  $\square$

**Corollary 9.5.** *The first term can be expressed in closed form as follows:*

$$(I) = \frac{(2g-2)!}{2^g (g-1)!} a_{g-1}^{(g-1)} - (-1)^g 2^{2-2g} (2g-3)! a_0^{(g-2)} \quad (29)$$

$$\times \left( \text{the coefficient of } x^{2g-1} \text{ in } \left( \frac{2x}{1-e^{-2x}} - 1 - x \right) \frac{e^x - 1}{x} \right)$$

*Proof.* Indeed, we recall that by definition  $\sum_{k=0}^{\infty} b_k (2x)^k = \frac{2x}{1-e^{-2x}}$ . If we subtract from this power series the linear part, i.e. subtract  $1 + x$ , and

multiply the result by  $\frac{e^x-1}{x} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$ , the coefficient of  $x^{2g-1}$  in the product is exactly equal to  $2^{2-2g}$  times the sum in the right-hand-side of the proposition.  $\square$

Combining this with the computation of terms (II) and (III) in the previous section, we see that the only non-zero  $a_N^{(g)}$  in our range is the one for  $N = 2g - 1$ , and the expression for  $a_{2g-1}^{(g)}$ , being the sum of three terms (I), (II), and (III), is as claimed.

## 10 Numerical examples

Here we list the numerical values of the terms (I),(II) and (III) as well as  $a_{2g-1}^{(g)}$  for small values of  $g$  (obtained by evaluating our formulas using Maple)

genus \ term	(I)	(II)	(III)	$a_{2g-1}^{(g)}$
2	$\frac{1}{12}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{11}{12}$
3	$-\frac{1}{80}$	$-\frac{25}{24}$	$\frac{5}{24}$	$-\frac{203}{240}$
4	$\frac{1}{672}$	$-\frac{49}{80}$	$\frac{7}{80}$	$-\frac{1759}{3360}$
5	$-\frac{1}{1296}$	$-\frac{3637}{2520}$	$\frac{1063}{7560}$	$-\frac{59123}{45360}$
6	$\frac{1}{220}$	$-\frac{23837}{315}$	$\frac{1639}{315}$	$-\frac{976649}{13860}$
7	$-\frac{11}{18}$	$-\frac{4194073}{189}$	$\frac{17594928013}{16329600}$	$-\frac{49254708341}{2332800}$

(30)

Note that the terms for  $g = 2, 3$  and  $4$  coincide with [vdG1, Table 2b], [vdG1, Table 3d] and [EGH, Theorem 1.1], respectively. For the last number we have to take an extra factor of  $1/2$  into account for  $g = 4$  since the computation in [EGH] was done for varieties rather than stacks. The values for  $g > 4$  are new.

## 11 Further comments

Throughout the paper we have worked with the perfect cone compactification  $\mathcal{A}_g^{\text{Perf}}$  of  $\mathcal{A}_g$ . In this case the Picard group has two generators, namely the Hodge line bundle  $L$  and the boundary  $D$ . In particular, the boundary is irreducible and a Cartier divisor (on the stack). There are other

toroidal compactifications, such as the Voronoi compactification  $\mathcal{A}_g^{\text{Vor}}$  and the Igusa compactification  $\mathcal{A}_g^{\text{Igu}}$ . Their Picard groups are much more complicated. Nevertheless, our computations are also relevant for other toroidal compactifications.

In general, a toroidal compactification is determined by the choice of an admissible fan  $\Sigma(F)$  for each cusp  $F$ . These choices must be made in such a way that the resulting collection of fans  $\tilde{\Sigma} = \{\Sigma(F)\}$  fulfills certain compatibility conditions. In the case of  $\mathcal{A}_g$  there exists exactly one cusp in each dimension, corresponding to the strata of the Satake compactification  $\mathcal{A}_g^{\text{Sat}}$  (cf. (5)). This means that, due to the compatibility conditions, a toroidal compactification of  $\mathcal{A}_g$  is already determined by the choice of an admissible fan in the cone  $\text{Sym}^{\geq 0}(g, \mathbb{R})$  of semi-positive definite real symmetric  $g \times g$ -matrices. Given any toroidal compactification  $\overline{\mathcal{A}}_g$ , the geometry of the partial compactification  $\overline{\mathcal{A}}_g \setminus \beta_k$  is determined by the intersection of the fan  $\Sigma$  with  $\text{Sym}^{\geq 0}(k-1, \mathbb{R})$  where we can think of elements in  $\text{Sym}^{\geq 0}(k-1, \mathbb{R})$  as  $(g \times g)$ -matrices by putting the matrices in the left hand corner and extending this by zeroes in the other entries. All admissible fans in  $\text{Sym}^{\geq 0}(g, \mathbb{R})$  coincide for  $g \leq 3$  (of course, up to taking a subdivision of this fan, which will result in a blow-up of the toroidal compactification, but we will disregard this), in particular  $\overline{\mathcal{A}}_g \setminus \beta_3 = \mathcal{A}_g^{\text{Perf}} \setminus \beta_3$ . As all our computations happen outside  $\beta_3$ , they apply to all toroidal compactifications (disregarding artificial blow-ups), where the meaning of  $D$  is that it is the closure of the boundary of Mumford's partial compactification.

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